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Comment

Relation between the thermodynamic Casimir effect in Bose-gas slabs and critical Casimir forces

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Abstract. – In a recent letter, Martin and Zagrebnov [*Europhys. Lett.,* **73** (2006) 1] discussed the *thermodynamic* Casimir effect for the ideal Bose gas confined in a thin film. We point out that their findings can be expressed in terms of previous general results for the Casimir effect induced by confined *critical fluctuations*. This highlights the links between the Casimir effect in the contexts of critical phenomena and Bose-Einstein condensation.

The ideal Bose gas undergoes a phase transition in the grand canonical ensemble if the chemical potential $\mu \leq 0$ equals its critical value $\mu_c = 0$ at which Bose-Einstein condensation takes place. This transition is accompained by density fluctuations with an increasing correlation length $\xi_+(\mu \to \mu_c) \sim (\mu_c - \mu)^{-\nu}$ ($\nu = \frac{1}{2}$ for the ideal gas).

In ref. [1] the authors consider the limit of large film thicknesses d and *fixed* μ of the grand canonical potential per unit (transverse) area $\varphi_d(T, \mu)$ of an ideal Bose gas in spatial dimension $D = 3$. For $\mu \neq \mu_c$ this implies that eventually $d/\xi_+ \gg 1$ and therefore the confining boundaries are subject to a vanishing Casimir force resulting from correlated fluctuations. On the other hand, at the critical point $\mu = \mu_c$, $d/\xi_+ \to 0$ and the fluctuations become long-ranged, giving rise to a Casimir force per unit area $F(d, T, \mu = \mu_c) = 2k_\mathrm{B}T\Delta/d^3 + \ldots$, characterized by a *universal* amplitude Δ(1). The cross over between these two regimes, which has not been investigated in ref. [1], is determined by $\delta\varphi_d(T, \mu) \equiv \varphi_d(T, \mu) - d\,\varphi_{bulk}(T, \mu) - \varphi_{surf}(T, \mu)$ (see eqs. (2) and (9) in ref. [1]) for large d and *fixed* ratio $d/\xi_+ \sim d(-\mu)^{\frac{1}{2}}$. Here we consider the case of Dirichlet boundary conditions (BCs), although the extention to other cases is straightforward. According to eq. (17) in ref. [1] one has:

$$-\frac{(2\pi)^{\frac{3}{2}}}{2}\beta d^2 \delta\varphi_d(T, \mu) = \left(\frac{d}{\lambda}\right)^3 \sum_{n,r=1}^{\infty} \frac{\mathrm{e}^{\beta\mu r}}{r^{5/2}} \mathrm{e}^{-2(nd/\lambda)^2/r} = \sum_{n=1}^{\infty} \left(\frac{\lambda}{d}\right)^2 \sum_{r=1}^{\infty} \Phi_n((\lambda/d)^2 r, u), \quad (1)$$

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(1)In passing we note that the amplitudes $\Delta(T)$ introduced in eqs. (3), (12), and (13) in ref. [1] are not *universal*, unless they are properly normalized as $\Delta \equiv \Delta(T)/k_\mathrm{B}T$.

where $\beta \equiv 1/(k_B T)$, $\lambda \equiv \hbar\sqrt{\beta/m}$ is the thermal wavelength, $\Phi_n(s,u) \equiv s^{-\frac{5}{2}}\mathrm{e}^{-u^2 s/2 - 2n^2/s}$, and $u \equiv (-2\beta\mu)^{\frac{1}{2}} d/\lambda \sim d/\xi_+$. For $d/\lambda \gg 1$, $(\lambda/d)^2 \sum_r \Phi_n((\lambda/d)^2 r, u) \mapsto \int_0^\infty \mathrm{d}s\, \Phi_n(s,u)$, with corrections decaying faster than any power of λ/d, leading to [2]

$$\beta d^2 \delta\varphi_d(T,\mu) = -\frac{1}{8\pi}\sum_{n=1}^{\infty} \frac{1+2u\,n}{n^3}\mathrm{e}^{-2u\,n} \equiv \Theta(u), \tag{2}$$

so that $F(d,T,\mu) = -\partial\delta\varphi_d(T,\mu)/\partial d = k_B T[2\Theta(u) - u\Theta'(u)]/d^3$. $\Theta(u)$ is a continuous, negative, and monotonicly increasing function of u providing the interpolation between the two cases discussed in ref. [1]: At the transition point $u = 0$ one recovers eq. (13) therein, *i.e.*, $\Delta \equiv \Theta(0) = -\zeta(3)/(8\pi)$. Due to $\Theta(u \gg 1) \sim \mathrm{e}^{-2u}$ for $d \gg \lambda$ and fixed μ we recover eq. (21): $|\delta\varphi_d(T,\mu)| \leq O(\mathrm{e}^{-\sqrt{-8\beta\mu}d/\lambda})$. It is easy to verify that $\Theta(u) = \Theta^{(1)}_{+O,O}(u)$ in $D = 3$ and with $N = 2$, where $\Theta^{(1)}_{+O,O}$ is given by eq. (6.6)(2) in ref. [3]. $\Theta^{(1)}_{+O,O}$ is the Gaussian, (1), *universal* scaling function above the bulk critical temperature, +, for the finite-size contribution $\beta d^{D-1}\delta\mathcal{F}_d(T)$ to the free energy $\mathcal{F}_d(T) = d\mathcal{F}_{bulk}(T) + \mathcal{F}_{surf}(T) + \delta\mathcal{F}_d(T)$ per unit (transverse) area of systems the critical properties of which are captured by the $O(N)$ symmetric Landau-Ginzburg (LG) Hamiltonian [4], confined in a $d \times \infty^{D-1}$ slab with Dirichlet BCs, O, O. This unnoticed connection between the results of refs. [3] and [1] holds also for periodic and Neumann BCs considered in ref. [1]. It is rooted in the fact that the grand canonical partition function for a weakly interacting Bose gas in D dimensions can be expressed, close to the transition point ($D > 2$), as a functional integral with weight $\mathrm{e}^{-\mathcal{S}[\phi]}$ (see, *e.g.*, refs. [4,5]) where $\phi(x)$ is a two-component real field, $\mathcal{S}$ the $O(2)$ LG Hamiltonian

$$\mathcal{S}[\phi] = \int \mathrm{d}^D x \left\{ \frac{1}{2}[\nabla\phi(x)]^2 + \frac{1}{2}r\phi^2(x) + \frac{g}{4!}[\phi^2(x)]^2 \right\}, \tag{3}$$

$r = -2m\mu/\hbar^2$, and $g = 48\pi a\hbar^4/\lambda^2$ where a is the scattering length. For the ideal gas $g = 0$, and $\mathcal{S}$ reduces to the so-called Gaussian model (defined only for $r > 0$), characterized by $\xi_+ = r^{-\frac{1}{2}}$. $\mathcal{S}$ is a particular case of the more general $O(N)$-symmetric LG Hamiltonian considered in ref. [3], where the *universal* scaling functions $\Theta(y_+)$ corresponding to different BCs (*surface universality classes*) imposed on $\phi(x)$ have been determined analytically within the field-theoretical ϵ-expansioon ($\epsilon = 4 - D$), as functions of the scaling variable $y_+ \equiv d/\xi_+$. For the ideal Bose gas $y_+ = dr^{\frac{1}{2}} = u$, as in eq. (2). Thus, the general results of ref. [3] predict also the Casimir amplitude Δ_{int} in a *weakly interacting* Bose gas (as well as for the superfluid transition, belonging to the same XY universality class $N = 2$). In $D = 3$ and for Dirichlet BCs $\Delta_{\mathrm{int}} \simeq -0.022$ compared to the ideal gas case $\Delta \simeq -0.048$.

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(2)The correct lower integration limit is $x = 1$ in accordance with eq. (6.5).